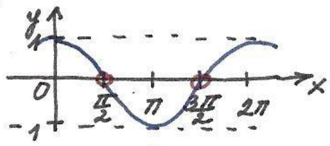
8. Základní goniometrické vzorce

– z def. gon. fcí obec. úhlu

① $\operatorname{tg} x = \dfrac{\sin x}{\cos x}$

$\mathcal{D}_{tg}$: $\cos x \neq 0$



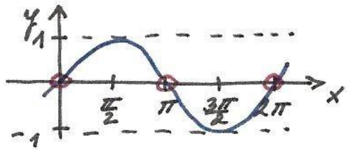
$x \neq \frac{\pi}{2} + k\pi$

$x \neq \frac{\pi}{2}(1+2k)$ liché nás. $\frac{\pi}{2}$

$\mathcal{D}_{tg} = R - \bigcup\limits_{k\in Z} \{\frac{\pi}{2}(2k+1)\}$

② $\operatorname{cotg} x = \dfrac{\cos x}{\sin x}$

$\mathcal{D}_{cotg}$: $\sin x \neq 0$



$x \neq 0 + k\pi$

$x \neq k\pi$ násobky π

$\mathcal{D}_{cotg} = R - \bigcup\limits_{k\in Z} \{k\pi\}$

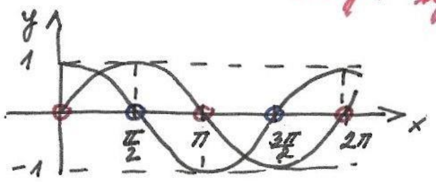
③ $\operatorname{tg} x \cdot \operatorname{cotg} x = 1$ pro $\forall x \in R - \bigcup\limits_{k\in Z} \{k\frac{\pi}{2}\}$ $\Rightarrow$ $\operatorname{tg} x = \dfrac{1}{\operatorname{cotg} x}$, $\operatorname{cotg} x = \dfrac{1}{\operatorname{tg} x}$

důkaz:

$\operatorname{tg} x \cdot \operatorname{cotg} x = \dfrac{\sin x}{\cos x} \cdot \dfrac{\cos x}{\sin x} = 1$

$\mathcal{D}$: $\cos x \neq 0 \wedge \sin x \neq 0$

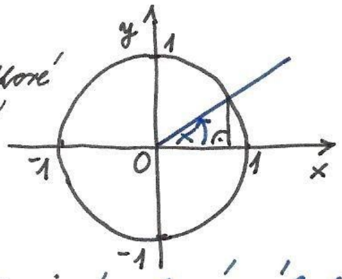
$\mathcal{D} = R - \bigcup\limits_{k\in Z} \{k\frac{\pi}{2}\}$

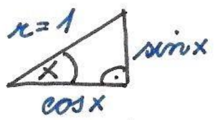


$x \neq k\frac{\pi}{2} \quad k \in Z$

④ $\forall x \in R$: $\sin^2 x + \cos^2 x = 1$

důkaz:
v jednotkové kružnici





z Pyth. věty
$\sin^2 x + \cos^2 x = 1$

– vzorce pro dvojnásobný úhel

⑤ $\forall x \in R$: $\sin 2x = 2 \sin x \cdot \cos x$

⑥ $\forall x \in R$: $\cos 2x = \cos^2 x - \sin^2 x$

[důkazy pomocí součtových vzorců – viz dále]

Příklady

① Urči hodnoty zbývajících goniometrických funkcí v bodě x, je-li

a) $\sin x = -\frac{1}{5}$ a $x \in (\pi, \frac{3}{2}\pi)$ III. kv.

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$|\cos x| = \sqrt{1-\sin^2 x} = \sqrt{1-(-\frac{1}{5})^2} =$

$= \sqrt{1-\frac{1}{25}} = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}$

III. kv. $\cos x < 0 \Rightarrow \cos x = -\frac{2\sqrt{6}}{5}$

$\operatorname{tg} x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{1}{5}}{-\frac{2\sqrt{6}}{5}} = \dfrac{5}{5\cdot 2\sqrt{6}} = \dfrac{1}{2\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}}$

$\operatorname{tg} x = \dfrac{\sqrt{6}}{12}$

$\operatorname{cotg} x = \dfrac{\cos x}{\sin x} = \dfrac{-\frac{2\sqrt{6}}{5}}{-\frac{1}{5}} = 2\sqrt{6}$

[nebo $\operatorname{cotg} x = \frac{1}{\operatorname{tg} x}$]

b) $\mathrm{cotg}\,x = -\frac{8}{15}$ $\quad \frac{\pi}{2} < x < \pi$ II. kv.

$\mathrm{tg}\,x = \frac{1}{\mathrm{cotg}\,x} = -\frac{15}{8}$

$\sin^2 x + \cos^2 x = 1 \quad | : \sin^2 x$

$1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$1 + \mathrm{cotg}^2 x = \frac{1}{\sin^2 x} \quad | \cdot \sin^2 x$

$\sin^2 x + \sin^2 x \cdot \mathrm{cotg}^2 x = 1$

$\sin^2 x (1 + \mathrm{cotg}^2 x) = 1$

$\sin^2 x = \frac{1}{1+\mathrm{cotg}^2 x}$

$|\sin x| = \sqrt{\frac{1}{1+\mathrm{cotg}^2 x}}$

$|\sin x| = \sqrt{\frac{1}{1+(-\frac{8}{15})^2}} = \sqrt{\frac{1}{1+\frac{64}{225}}} = \sqrt{\frac{1}{\frac{289}{225}}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

$x \in$ II. kv. $\oplus$ $\quad \sin x = \frac{15}{17}$

$\mathrm{cotg}\,x = \frac{\cos x}{\sin x}$

$\cos x = \mathrm{cotg}\,x \cdot \sin x = -\frac{8}{15} \cdot \frac{15}{17}$

$\cos x = -\frac{8}{17}$

nebo [$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$|\cos x| = \sqrt{1-\sin^2 x}$

$|\cos x| = \sqrt{1-\frac{225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17}$

II. kv. $\ominus$ $\quad \cos x = -\frac{8}{17}$]

c) $\mathrm{cotg}\,x = 1$ $\quad \sin x = -\frac{\sqrt{2}}{2}$

$\oplus$ I., III. kv. $\quad \ominus$ III., IV. kv.

III. kv.

$\mathrm{tg}\,x = \frac{1}{\mathrm{cotg}\,x} = \frac{1}{1} = 1$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$|\cos x| = \sqrt{1-(-\frac{\sqrt{2}}{2})^2} = \sqrt{1-\frac{2}{4}} = \sqrt{\frac{2}{4}}$

$|\cos x| = \frac{\sqrt{2}}{2}$

III. kv. $\ominus$ $\quad \cos x = -\frac{\sqrt{2}}{2}$

② Vypočítejte hodnoty zbýv. goniom. fcí v bodě x_0, jl-li

DÚ a) $\cos x = \frac{4}{5}$ $\quad x \in (\frac{3\pi}{2}, 2\pi)$ IV. kv.

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$|\sin x| = \sqrt{1-\cos^2 x}$

$|\sin x| = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

IV. kv. $\ominus$ $\quad \sin x = -\frac{3}{5}$

$\mathrm{tg}\,x = \frac{\sin x}{\cos x} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3 \cdot 5}{4 \cdot 5} = -\frac{3}{4}$

$\mathrm{cotg}\,x = \frac{1}{\mathrm{tg}\,x} = -\frac{4}{3}$

[nebo $\mathrm{cotg}\,x = \frac{\cos x}{\sin x} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$]

b) $\mathrm{tg}\,x = -2\sqrt{2}$ $\quad x \in (\frac{\pi}{2}, \pi)$ II. kv.

$\mathrm{cotg}\,x = \frac{1}{\mathrm{tg}\,x}$

$\mathrm{cotg}\,x = \frac{1}{-2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2 \cdot 2} = -\frac{\sqrt{2}}{4}$

$\sin^2 x + \cos^2 x = 1 \quad | : \cos^2 x$

$\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$

$\mathrm{tg}^2 x + 1 = \frac{1}{\cos^2 x} \quad | \cdot \cos^2 x$

$\cos^2 x \cdot \mathrm{tg}^2 x + \cos^2 x = 1$

$\cos^2 x (\mathrm{tg}^2 x + 1) = 1$

$\cos^2 x = \frac{1}{\mathrm{tg}^2 x + 1}$

$|\cos x| = \sqrt{\frac{1}{\mathrm{tg}^2 x + 1}}$

$|\cos x| = \sqrt{\frac{1}{(-2\sqrt{2})^2+1}} = \sqrt{\frac{1}{4 \cdot 2+1}} = \sqrt{\frac{1}{9}}$

$|\cos x| = \frac{1}{3}$

II. kv. $\ominus$ $\quad \cos x = -\frac{1}{3}$

$tg\,x = \dfrac{\sin x}{\cos x}$

$\sin x = \cos x \cdot tg\,x$

$\sin x = -\dfrac{1}{3} \cdot (-2\sqrt{2})$

$\sin x = \dfrac{2\sqrt{2}}{3}$

$\sin^2 x + \cos^2 x = 1$

nebo $\sin^2 x = 1 - \cos^2 x$

$|\sin x| = \sqrt{1-\cos^2 x}$

$|\sin x| = \sqrt{1-\dfrac{1}{9}} = \sqrt{\dfrac{8}{9}} = \dfrac{2\sqrt{2}}{3}$

II. kv. (+) $\sin x = \dfrac{2\sqrt{2}}{3}$

③ Pro která $x \in R$ je výraz definován a zjednodušte ho

a) $\dfrac{1+tg^2 x}{1+cotg^2 x}$ (1. zp.) $= \dfrac{1+tg^2 x}{1+\frac{1}{tg^2 x}} = \dfrac{1+tg^2 x}{\frac{tg^2 x+1}{tg^2 x}} = \dfrac{(1+tg^2 x)\,tg^2 x}{1+tg^2 x} = tg^2 x$ $\left[cotg\,x = \dfrac{1}{tg\,x}\right]$

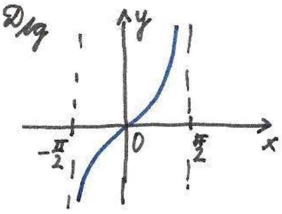
(2. zp.) $= \dfrac{1+\frac{1}{cotg^2 x}}{1+cotg^2 x} = \dfrac{\frac{cotg^2 x+1}{cotg^2 x}}{1+cotg^2 x} = \dfrac{cotg^2 x+1}{cotg^2 x\,(1+cotg^2 x)} = \dfrac{1}{cotg^2 x} = tg^2 x$ $\left[tg\,x = \dfrac{1}{cotg\,x}\right]$

(3. zp.) $= \dfrac{1+\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\cos^2 x}{\sin^2 x}} = \dfrac{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}} = \dfrac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \dfrac{\sin^2 x}{\cos^2 x} = tg^2 x$

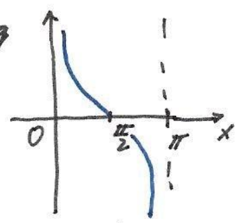
$\left[tg\,x = \dfrac{\sin x}{\cos x} \quad cotg\,x = \dfrac{\cos x}{\sin x} \quad \sin^2 x + \cos^2 x = 1\right]$

určení $\mathcal{D}$: $\mathcal{D} = \mathcal{D}_{tg} \cap \mathcal{D}_{cotg} \wedge$ jmen. $\neq 0$

(1. zp.) $\mathcal{D}_{tg}$



$\mathcal{D}_{cotg}$



$x \neq \frac{\pi}{2} + k\pi$

$x \neq \frac{\pi}{2}(1+2k)$

liché nás. $\frac{\pi}{2}$

$x \neq \pi + k\pi \neq 0 + k\pi$

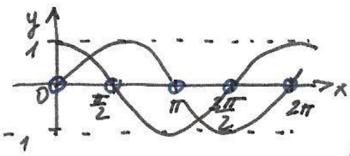
$x \neq k\pi \neq \frac{\pi}{2} \cdot 2k$

sudé nás. $\frac{\pi}{2}$

$\mathcal{D}_{tg} \cap \mathcal{D}_{cotg}$: $x \neq k\frac{\pi}{2}$ násobky $\frac{\pi}{2}$

nebo $tg\,x = \dfrac{\sin x}{\cos x}$ $\quad cotg\,x = \dfrac{\cos x}{\sin x}$

(2. zp.) $\cos x \neq 0 \wedge \sin x \neq 0$



$x \neq k\frac{\pi}{2}$

$\mathcal{D}_{tg} \cap \mathcal{D}_{cotg}$

+ jmen. zl. $\neq 0$

$cotg^2 x \neq -1$

platí pro $\forall x \in R$

$\mathcal{D} = R - \bigcup_{k \in Z} \{k\frac{\pi}{2}\}$

b) $\dfrac{\cos^2 x}{1+\sin x} = \dfrac{1-\sin^2 x}{1+\sin x} = \dfrac{(1-\sin x)(1+\sin x)}{1+\sin x} = 1-\sin x$

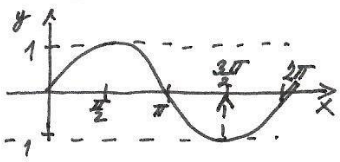
$a^2 - b^2 \qquad (a-b)\cdot(a+b)$

$[\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x]$

$\mathcal{D}$: $1+\sin x \neq 0$

$\sin x \neq -1$

$x \neq \frac{3\pi}{2} + 2k\pi$



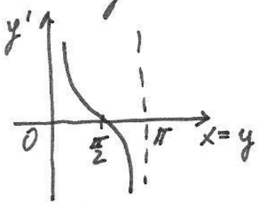
$\mathcal{D} = R - \bigcup_{k \in Z} \{\frac{3\pi}{2} + 2k\pi\}$

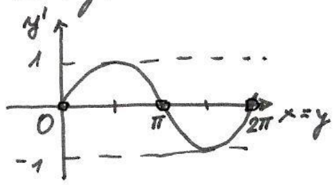
c) $1-\sin^2 y+\operatorname{cotg}^2 y\cdot\sin^2 y = 1-\sin^2 y+\frac{\cos^2 y}{\sin^2 y}\cdot\sin^2 y = \underbrace{1-\sin^2 y}_{\cos^2 y=1-\sin^2 y}+\cos^2 y =$

$= \cos^2 y+\cos^2 y = 2\cos^2 y$

$[\sin^2 y+\cos^2 y=1]$

D: Dcotg nebo $\sin y\neq 0$





$y\neq 0+k\pi$
$y\neq k\pi$

$y\neq 0+k\pi$
$y\neq k\pi$

$D = R-\bigcup_{k\in Z}\{k\pi\}$

d) DÚ $\frac{\sin x-\sin^3 x}{\cos x-\cos^3 x} = \frac{\sin x(1-\sin^2 x)}{\cos x(1-\cos^2 x)} = \frac{\sin x\cdot\cos^2 x}{\cos x\cdot\sin^2 x} = \frac{\cancel{\sin x}\cdot\cos x}{\cancel{\cos x}\cdot\sin x} = \operatorname{tg} x$

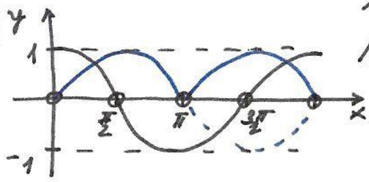
($1-\sin^2 x = \cos^2 x$, $1-\cos^2 x = \sin^2 x$)

$\cos x\neq 0 \wedge 1-\cos^2 x\neq 0$

$\sin^2 x\neq 0$ nebo $\rightarrow \sin x\cdot\sin x\neq 0$

$\sin x\neq 0$

$|\sin x|\neq 0$



$x\neq\frac{\pi}{2}\cdot k$

$D = R-\bigcup_{k\in Z}\{k\frac{\pi}{2}\}$

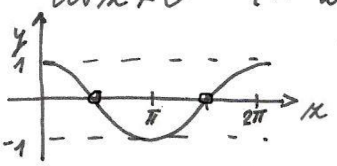
e) DÚ $\frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{\frac{\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}} = \frac{\sin x\cdot\cos^2 x}{\cos x\cdot 1} = \sin x\cdot\cos x$

($\cos^2 x+\sin^2 x = 1$)

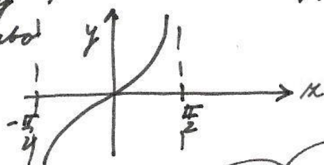
D: $\cos x\neq 0$ (= Dtg) $\wedge$ $1+\operatorname{tg}^2 x\neq 0$

$\operatorname{tg}^2 x\neq -1$

$|\operatorname{tg} x|\neq -1$ nikdy



nebo



$x\neq\frac{\pi}{2}+k\pi$
$x\neq\frac{\pi}{2}(2k+1)$

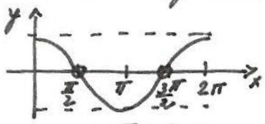
$D = R-\bigcup_{k\in Z}\{(2k+1)\frac{\pi}{2}\}$

4) Upravte, udejte podmínky

a) $\frac{\sin 2x}{\cos^2 x} = \frac{2\sin x\cdot\cancel{\cos x}}{\cos^2 x} = \frac{2\cdot\sin x}{\cos x} = 2\operatorname{tg} x$

$[\sin 2x = 2\sin x\cdot\cos x]$

D: $\cos x\neq 0$ (jmen. $\neq 0$)



$x\neq\frac{\pi}{2}+k\pi$
$x\neq\frac{\pi}{2}(1+2k)$

$D = R-\bigcup_{k\in Z}\{\frac{\pi}{2}+k\pi\}$

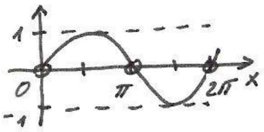
$D = R-\bigcup\{\frac{\pi}{2}(2k+1)\}$

b) $\dfrac{\sin 2x}{1-\cos 2x} = \dfrac{2\sin x\cdot\cos x}{1-(\cos^2 x-\sin^2 x)} = \dfrac{2\sin x\cdot\cos x}{\underbrace{1-\cos^2 x}_{\sin^2 x}+\sin^2 x} = \dfrac{2\sin x\cdot\cos x}{\sin^2 x+\sin^2 x} =$

$[\cos 2x = \cos^2 x-\sin^2 x]$ $\quad$ $[\sin^2 x+\cos^2 x=1]$

$= \dfrac{\cancel{2}\cancel{\sin x}\cdot\cos x}{2\sin^2 x} = \dfrac{\cos x}{\sin x} = \cot g\, x$ $\quad$ $\left[\dfrac{\cos x}{\sin x} = \cot g\, x\right]$

$\mathcal{D}:\ 1-\cos 2x \neq 0 \iff \sin x \neq 0$



$x \neq 0 + k\pi$
$x \neq k\pi$

$\mathcal{D} = R - \bigcup_{k\in Z}\{k\pi\}$

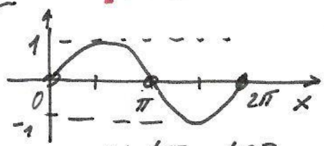
c) $\dfrac{1+\cos 2x}{1-\cos 2x} = \dfrac{1+\cos^2 x-\sin^2 x}{1-(\cos^2 x-\sin^2 x)} = \dfrac{1-\sin^2 x+\cos^2 x}{1-\cos^2 x+\sin^2 x} = \dfrac{\cos^2 x+\cos^2 x}{\sin^2 x+\sin^2 x} =$

$= \dfrac{2\cos^2 x}{2\sin^2 x} = \dfrac{\cos^2 x}{\sin^2 x} = \cot g^2 x$

$\left[\begin{array}{l}\cos 2x = \cos^2 x-\sin^2 x\\ \sin^2 x+\cos^2 x=1 \Rightarrow \cos^2 x = 1-\sin^2 x\\ \qquad\qquad\qquad\quad \sin^2 x = 1-\cos^2 x\end{array}\right]$

$\dfrac{\cos x}{\sin x} = \cot g\, x$

$\mathcal{D}:\ 1-\cos 2x\neq 0 \iff 2\sin^2 x\neq 0$
$\sin x \neq 0$



$x\neq k\pi \quad k\in Z$

$\mathcal{D} = R - \bigcup_{k\in Z}\{k\pi\}$

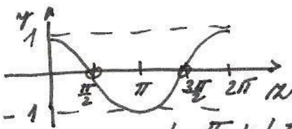
d) $\dfrac{1-tg^2 x}{\cos 2x} = \dfrac{1-\dfrac{\sin^2 x}{\cos^2 x}}{\cos^2 x-\sin^2 x} = \dfrac{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\cos^2 x-\sin^2 x} = \dfrac{\cancel{(\cos^2 x-\sin^2 x)}}{\cos^2 x\,\cancel{(\cos^2 x-\sin^2 x)}} = \dfrac{1}{\cos^2 x}$

DÚ

$\mathcal{D}:\ \cos 2x\neq 0$

! subst. $u = 2x$

$\cos u \neq 0$



$u \neq \frac{\pi}{2}+k\pi$

$u\neq \frac{\pi}{2}+k\pi$
zpět k subst.
$2x \neq \frac{\pi}{2}+k\pi$
$x\neq \frac{\pi}{4}+k\frac{\pi}{2}$

$\mathcal{D} = R - \bigcup_{k\in Z}\left\{\frac{\pi}{4}+k\frac{\pi}{2}\right\}$

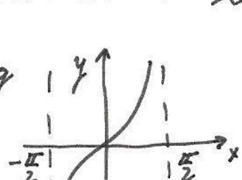
e) $\cos 2x + \sin 2x\cdot tg\,x = \cos^2 x-\sin^2 x + 2\sin x\cdot\cos x\cdot tg\,x =$

DÚ

$= \cos^2 x-\sin^2 x+2\sin x\cdot\cancel{\cos x}\cdot\dfrac{\sin x}{\cancel{\cos x}} = \cos^2 x-\sin^2 x+2\sin^2 x =$

$= \cos^2 x+\sin^2 x = 1$

$\mathcal{D} = \mathcal{D}_{tg}$



$x\neq \frac{\pi}{2}+k\pi$

$\mathcal{D} = R - \bigcup_{k\in Z}\left\{\frac{\pi}{2}+k\pi\right\}$

$\mathcal{D} = R - \bigcup_{k\in Z}\left\{\frac{\pi}{2}(2k+1)\right\}$

⑤ Uprav, urči definiční obory

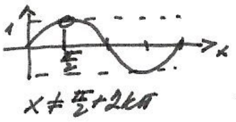
DÚ

a) $(\cos y+\sin y)^2+(\cos y-\sin y)^2 = \cos^2 y + 2\sin y\cdot\cos y + \sin^2 y +$
$+ \cos^2 y - 2\sin y\cdot\cos y + \sin^2 y = 1+1 = 2$ $\quad D = R$

b) $\dfrac{\cos^2 x}{1-\sin x} = \dfrac{1-\sin^2 x}{1-\sin x} = \dfrac{(1-\sin x)(1+\sin x)}{1-\sin x} = 1+\sin x$

[$\cos^2 x = 1-\sin^2 x$
$\sin^2 x+\cos^2 x = 1$]

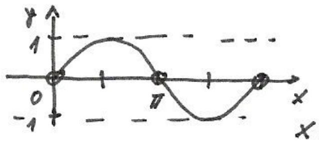
[D: $1-\sin x \neq 0$
$\sin x \neq 1$



$x \neq \frac{\pi}{2}+2k\pi$] $\quad D = R - \bigcup_{k\in Z}\{\frac{\pi}{2}+2k\pi\}$

c) $\dfrac{\cos^2 x\cdot \mathrm{tg}\, x}{1-\cos^2 x} = \dfrac{\cos^2 x\cdot\frac{\sin x}{\cos x}}{\sin^2 x} = \dfrac{\cos x\cdot\sin x}{\sin^2 x} = \dfrac{\cos x}{\sin x} = \mathrm{cotg}\, x$

[D: $\sin x \neq 0$



$D = R - \bigcup_{k\in Z}\{k\pi\}$

$x \neq 0 + k\pi$
$x \neq k\pi$]